Joint Estimation of the Activity and the Events of Interaction in SPECT Systems

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***Abstract*—We describe a unified probabilistic graphical model for joint reconstruction of the activity and of the coordinates of the events of interaction in Emission Imaging. This new formulation is a discrete-discrete alternative to the continuous-discrete List Mode Likelihood model described by Barrett et al. The discrete-discrete formulation allows us to account for non-homogeneous resolution of the gamma-camera while adopting computationally efficient shift-invariant projection and back-projection algorithms. We describe an application of the model for the compensation of disrupted photo-detectors is SPECT.**

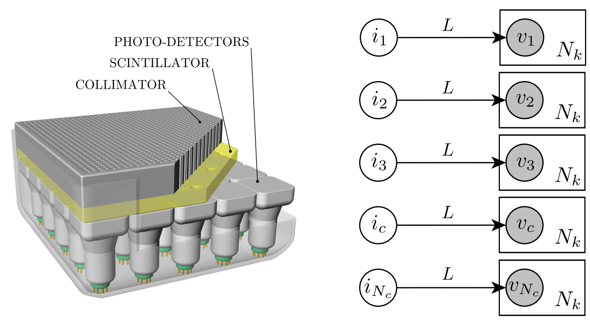

Fig. 1. Left: monolithic gamma camera. Right: Bayesian Network (directed acyclical graph) representing the model of the gamma-camera. The location of interaction i_c of each photon c is considered the unique cause of the measurement vector v_c. N_k is the number of photo-detectors. The structure of the model (d-separation of the graph) corresponds with the model in equation (1).

I. Introduction

We introduce a new discrete computational model that allows us to account for the spatial dependence of the resolution of the gamma-camera and for the depth-dependent system response. This formulation yields a computationally efficient reconstruction algorithm that accounts for non-uniform resolution of the gamma-camera.

The algorithm is applied to the reconstruction of synthetic perfusion SPECT image. In order to show the efficacy of the algorithm, high non-uniformity of the resolution is enforced by simulating a disrupted photo-detector.

II. Methods

We make use of Bayesian Networks in order to describe the discrete-discrete probabilistic model of the emission imaging system and to derive an optimisation algorithm for joint reconstruction of the activity and of the events of interaction.

A. Model of the gamma-camera

The model of the gamma camera described here refers to the monolithic gamma camera as from the original design of Anger (see figure 1-left), though the model may be applied to other configurations of the gamma camera.

The gamma-camera, in response to the interaction of a gamma photon, generates electrical signals correlated to the position of the interaction and to the energy of the gamma photon. By employing a probabilistic model of the camera response, one may infer the location of each event of interaction and the energy of the detected gamma photon. Indexing the photon interactions with $c = \{1, \dots, N_c\}$, each interaction generates a vector of electrical signals v_c of length N_k, where N_k is the number of secondary photo-detectors. Discretising the volume of the scintillator crystal in N_d voxels indexed with $d = \{1, \dots, N_d\}$, let us indicate with l_{dk} the probability that a secondary photon emitted in consequence of a gamma-interaction in d is detected by photo-detector k. $L = l_{dk}$ constitutes the system matrix of the gamma-camera, encompassing its geometry and optical characteristics. Assuming that the coordinates of interaction of photon c with the scintillator crystal are the unique underlying cause that determines the read-out signal v_c, the probabilistic model of the gamma-camera may be obtained with a *causal modelling* approach, by encoding the causality relation with a directed arrow of a Directed Acyclical Graph [**?**], reported in figure 1-right. The *d-separation* properties of the graph imply the following factorised conditional probability distribution:

$$p(v_c|i_c = e_d) = \prod_{k=1}^{N_k} \frac{e^{-l_k \cdot e_d^T} \left(l_k \cdot e_d^T\right)^{v_{ck}}}{v_{ck}!} \tag{1}$$

where the uncertainty associated to the measurement of secondary photons is considered to be characterised by the Poisson distribution (counting of the secondary photons), discarding electronic noise and other sources of uncertainty such as dark counts. e_d is the unit vector of lenght N_d with element d equal to 1 and $i_c \in \{e_1, e_2, \dots, e_d, e_{N_d}\}$ is the indicator function whose value indicates the location of interaction of photon c.

This model of the gamma camera if often employed for the reconstruction of the events of interaction [**?**]. It is often stated that the Maximum Likelihood Estimator (MLE) - the value of i_c that maximises the likelihhod function (1) is optimum, providing the most accurate measurement of the location of interaction. The MLE estimator is in fact optimum in the sense

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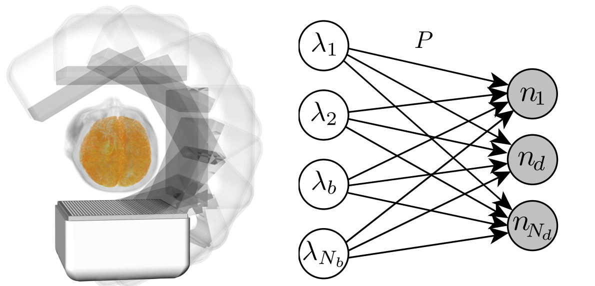

Fig. 2. Left: SPECT system. Right: directed acyclical graph of the SPECT system model. Radio-activity $\lambda = \{\lambda_1, \dots, \lambda_b, \dots, \lambda_{N_b}\}$ is considered the unique cause underlying the measurement of photon counts $n = \{n_1, \dots, n_d, \dots, N_d\}$. The structure of the graphical model (d-separation) corresponds to the factorised model in equation (2)

of minimum L2-norm of the expected error of the location of interaction (Cramer-Rao lower bound [**?**]), if one considers one event alone without concerning about the hidden cause of the gamma-emission and that many more measurements relate to that same underlying cause. If one could force the location of interactions to happen at a given location (x, y, z) of the scintillator crystal, the MLE would estimate the location of interaction with minimum error. However, considering the problem more carefully, there is a structure in the reconstruction problem in the case of the tomographic setup. As explained in the last paragraph of this section, the structure, exposed by the graphical model, can be exploited in order to obtain measurements of the locations of interaction with higher accuracy than predicted by the Cramer-Rao bound for the single event.

B. Emission and propagation of the gamma photons

Given a large number of detected photons, in SPECT (and PET), one estimates the spatial distribution of radio-activity. Shepp and Vardi introduced in 1982 a discrete computational model of the emission/detection of gamma photons based on the assumption that activity and locations of interaction are discrete [1]. Assuming that the locations of interaction are discrete, as one counts random (gamma interaction) events in each discrete location of interaction, such model is again expressed by a product of Poisson distributions. Assuming that the activity is a grid of point sources that emit events at rates $\lambda = \{\lambda_1, \dots, \lambda_b, \dots, \lambda_{N_b}\}$, and letting $P = p_{bd}$ be the probability that a gamma photon emitted in b *interacts* in d (encompassing the characteristics of the collimator, position of the camera and the attenuation map):

$$p(n|\lambda) = \prod_{d}^{N_d} \frac{e^{-\sum_b p_{bd}\lambda_b}\left(-\sum_b p_{bd}\lambda_b\right)^{n_d}}{n_d!} \tag{2}$$

$$n_d = \sum_{c=1}^{N_c} \hat{i}_c \cdot e_d^T \tag{3}$$

the product being implied again by the conditional independences of the model, represented graphically in figure 2-right. The activity is modelled as a grid of point processes that emit photons at a rate

The activity may be estimated by MLE, which consists in finding the value of λ that maximises (2). Optimisation of (2) may be obtained by the Expectation Maximisation algorithm described by Shepp and Vardi [1] or other optimisation algorithms. The optimisation of (2) with gradient-type algorithms such as MLEM implies iterative projection and back-projection in order to obtain estimates of λ that have increasing likelihood.

The model described in this section, identical to the formulation of Shepp and Vardi, is based on the assumption that the location of interaction of each photon can be measured exactly. This model may be extended in order to account for the uncertainty of the locations of interaction. The uncertainty about the location of interaction of the events can be taken into account in the reconstruction of the activity by assuming that the coordinates of interaction of a photon are normally distributed around its MLE estimate. The probability distribution of the location of interaction is in fact asymptotically (high number of secondary photons) normal, with variance equal to the inverse of the Fisher Information Matrix [2]. The uncertainty is however in general not uniform across the gamma-camera and normality only holds asymptotically. Simply by reformulating P_{bd} as the "probability that a photon emitted in b *is detected* in d" rather then the "probability that a photon emitted in b *interacts* in d", the most likely estimate that maximises (2) now accounts for the uncertainty associated with the locations of interaction. However this modification makes the projector and the back-projector shift-variant.

Barrett *et al.* [2] have introduced a continuous model (List-mode), which can be derived as the limit case of the discrete model for infinitely small detector bins. Increasing the number of detector bins, the algorithm produced by the continuous formulation becomes increasingly more computationally efficient than the discrete counterpart due to increasing zero entries in the binned photon counts [2]. While the List-mode model accounts for the uncertainty associated with the measurements of the locations of itneraction of the gamma photons, as described in [2] it is only practical to consider approximated probability distributions for the locations of interaction.

In the discrete-discrete formulation that we describe here, $P = p_{bd}$ expresses the "probability that a photon emitted in b *interacts* in d".

C. Joint model

D. Optimisation

We seek the activity and location of interaction of each photon that maximise the joint probability distribution.

The system is optimised by the Iterative Conditional Modes algorithm [3], iterating **i)** and **ii)**:

i) Estimation of the activity given the provisional estimate of the locations of interaction. A new estimate that is guaranteed to increase the the marginal probability distribution [1] is computed with the Expectation Maximisation algorithm:

$$\hat{\lambda}^{[m+1]} = \arg\max_{\lambda} \log p(\hat{n}^{[m+1]}|\lambda) \tag{4}$$

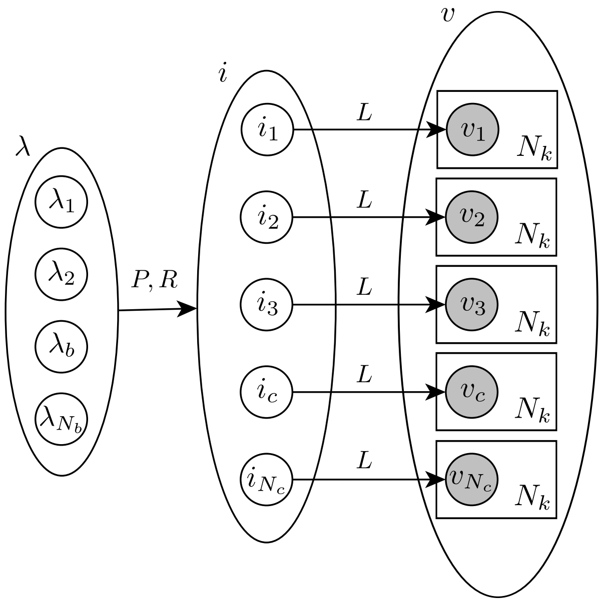

Fig. 3. Probabilistic graphical model (PGM) of the SPECT imaging system. Observed variables are shaded. Photon interactions are allowed at discrete locations indexed by $c = \{1, \ldots, N_c\}$. Activity λ determines N_c photon interactions, photons being indexed by $c = \{1, \ldots, N_c\}$. The location of interaction of each photon c determines the set of N_k electrical signals v_c. $P = p_{bd}$ describes the geometry of the system (collimator, attenuation map) and L the system model of the gamma camera, relating the allowed locations of interaction with the expected electrical signals.

$$p(\hat{n}^{[m+1]}|\lambda) = \prod_d^{N_d} \frac{e^{-\sum_b^{N_b} p_{bd}\lambda_b} \left(-\sum_b^{N_b} p_{bd}\lambda_b\right)^{\hat{n}^{[}_d m+1]}}{\hat{n}_d^{[m+1]}!} \quad (5)$$

$$\hat{n}_d^{[m+1]} = \sum_{c=1}^{N_c} \hat{i}_c^{[m+1]} \cdot e_d^T \quad (6)$$

where n_d is the number of photon counts at location d: $n_d = \sum_{n=1}^{N_c} i_n \cdot e_d$.
ii) Estimation of the locations of interaction given the provisional estimate of the activity.

$$\hat{i}^{[m+1]} = \arg\max_i \log p(i|\hat{\lambda}^{[m]}) p(v|i) \quad (7)$$

This factorises according to the conditional independence $i_c \perp i'_c|\lambda \forall c, c'$ (d-separation of the graphical model in figure 3):

$$\hat{i}^{[m+1]} = \arg\max_i \log p(i_c|\hat{\lambda}^{[m]}) p(v_c|i_c) \quad (8)$$

The conditional probability distribution $p(i_c = e_d|\hat{\lambda}^{[m]})$ has the following expression (see Appendix A):

$$p(i_c = e_d|\hat{\lambda}^{[m]}) = \frac{\sum_b^{N_b} p_b \cdot e_d^T \hat{\lambda}_b^{[m]}}{\sum_d^{N_d} \sum_b^{N_b} p_b \cdot e_d^T \hat{\lambda}_b^{[m]}} \quad (9)$$

and

$$p(v_c|i_c = e_d) = \prod_{k=1}^{N_k} \frac{e^{-l_k \cdot e_d^T} \left(l_k \cdot e_d^T\right)^{v_{ck}}}{v_{ck}!} \quad (10)$$

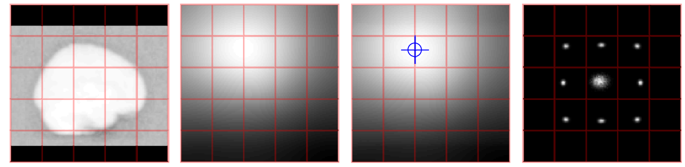

Fig. 4. From left to right: 1) log of the prior probability distributio of the location of interaction; 2) log likelihood of the location of interaction; 3) log posterior and location of the maximum a posteriori; 4) visualisation of the uncertainty in different locations of the detector obtained by reconstructing 10000 events for each location: the resolution in the centre is lower due to the broken photo-detector. In all images the red boxes represent the square photo-detectors.

The estimates of the coordinates of interaction $\hat{i}_c$ can be calculated by exhaustive search. The hierarchical grid search algorithm described in [4] for maximum likelihood estimation of the coordinates of interaction requires a small modification to account for 9.

III. Experiments

It is not uncommon that one or more photo-detectors of a gamma-camera stop working due to hardware failure. In this condition the resolution of the camera in the region near the disrupted detector is lower, as visualised in figure 4-right. We adopt the iterative algorithm described in the Methods section to estimate jointly the activity and the coordinates of interaction. A SPECT perfusion study is simulated with the SIMIND Monte Carlo simulator with $50M$ counts and ideal, infinite resolution gamma camera 5-left. Events of interaction are filtered based on simple energy thresholding, in order to reduce the effect of scattered photons. The gamma interactions are simulated according to the model **??** for an array of 25 square photo-detectors of size $4 \times 4cm$, quantum efficiency $\nu = 0.8$ and $1cm$ thick NaI(Tl) crystal with yield of 5000 photons. The same simulation is performed disabling the central photo-detector 5.

Joint reconstruction of the activity and the events of interaction with 100 iterations of the algorithm show, at visual inspection, recovery of the lost resolution. This is explained by the flow of information amongst the cameras through the iterative process. In the final version of the manuscript, quantitative assessment of the improvement in terms of bias variance of the activity estimates will be presented.

IV. Discussion

We introduce a new computational model of SPECT and an iterative algorithm for concurrent estimation of the activity and of the events of interaction. Such formulation allows us to account for the non-uniformity of the resolution of the gamma-camera employing efficient shift-invariant projection and back-projection algorithms. We have demonstrated how the algorithm can be used to compensate for broken photo-detectors. The model that we describe assumes that photons are unscattered, relying on energy thresholding for discarding scattered photons. Further development will extend the model to include the measurement of energy and scattered photons in a unified probabilistic framework.

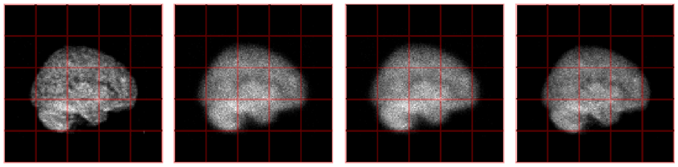

Fig. 5. From left to right: 1) ideal sinogram, obtained under the assumption of perfect reconstruction of the locations of interaction; 2) ML reconstruction of the interaction events; 3) ML reconstruction of the interaction events with broken photo-detector (central detector); 4) joint estimation of the activity and events of interaction, 100 iterations.

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